Analysis of spatial shear wall structures of variable cross-section

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Abstract

A method has been proposed for the analysis of three-dimensional shear wall and shear core assemblies with variable dimensions and geometries. The analysis is based on a variant of the continuum method. In the continuous approach the connecting beams and vertical joints are replaced by equivalent continuous connections. The differential equation systems for shear wall structure segments of constant cross-section are uncoupled by orthogonal eigenvectors. The solution matches the boundary conditions of the upper and lower part of the wall at the plane of contiguity, at which an abrupt change in cross-section occurs. This yields a system of linear equations for the determination of the constants of integration. The correctness and efficiency of the continuous connection method is illustrated by application of the technique to the analysis of spatial, complex wall system of variable cross-section.

Keywords: shear wall structures, variable cross-section, continuous connection method, tall buildings

1. Introduction

The application of continuum method to the analysis of coupled shear walls with abrupt changes in cross-section has been considered in Ref. [7], [8], [2], [9], [6], [4], [1], [11]. The analysis of three-dimensional shear wall systems, using the iterative technique based on a combination of the finite strip method and the continuum method, has been presented in Ref. [4]. In Ref. [5] discrete force method has been developed for the solution of such problems.

The purpose of this paper is to present the effective algorithm for the analysis of spatial shear wall structures of variable cross-section, using the variant of continuous connection method.

2. Governing differential equations

Equation formulations for a three-dimensional continuous model of the shear wall structure with the constant cross-section have been given in Ref. [10]. A structure, which changes its cross-section along the height, can be divided into \( n \) segments, each one being of constant cross-section. For \( k \)-th segment the differential equations can be stated as follows:

\[
z \in (h_{k-1}, h_k) \quad B_{(k)j} N_{(N(k))}^{(j)} f_{(k)j}(z) = A_{(k)j} N_{(N(k))}^{(j)} f_{(k)j}(z),
\]

where \( B_{(k)j} \) is \( n_e \times n_e \) diagonal matrix, containing continuous connection flexibilities, \( A_{(k)j} \) is \( n_e \times n_e \) symmetric, positive definite matrix, dependent on a structure, \( n_e \) is the number of continuous connections, \( N_{(N(k))}^{(j)}(z) \) is a vector containing unknown functions of the shear force intensity in continuous connections and \( f_{(k)j}(z) \) is a vector formed on the basis of given loads for the \( k \)-th segment of the shear wall structure.

The boundary conditions for the whole structure take the following form [11]:

\[
N_{(N(1))}(0) = -B_{(1)}^{-1} S_E^T z_0, \quad N_{(N(n))}(H) = 0,
\]

where \( S_E \) is \( n_e \times n_e \) boolean matrix, related to the interaction between shear walls and continuous connections, \( z_0 \) is the vector containing given settlements of shear walls, \( n_e \) is the number of shear walls, \( h_k \) is the ordinate of \( k \)-th change of the cross-section and \( H \) is the structure height.

In contemporary designs of tall buildings structures with significant changes in geometry occur, such as walls with openings missing on the lower floors or shear walls missing on the upper floors. In order to enable an accurate analysis of these difficult cases, the refined boundary conditions for shear force intensity functions at the plane of contiguity, at which an abrupt change in cross-section occurs, have been derived in the following form:

\[
N_{(N(k))}(h_k) = B_{(k)}^{-1} B_{(k+1)}^{(j)} N_{(N(k+1))}^{(j)}(h_k),
\]

where \( C_{(k)} \) is the \( 3n \times 3n \) matrix containing the coordinates of the points of contraflexure in the connecting beams in the local coordinate systems, \( L \) is the \( 3n \times n_e \) matrix of coordinates transformation from the global OXYZ system to the local systems, i.e. the systems of principal axes of shear walls, \( V(z) \) is the vector containing the functions of horizontal displacements of the structure, \( K_S \) is the \( n \times n \) diagonal matrix, \( K_S = \text{diag}(1/EA) \) and \( n_{(N(1))} \) is the vector containing the normal forces in shear walls.

It should be emphasized here that the mid-points of the connecting beams in different segments should lie on the same vertical line. The derivation of Eqn (3) and Eqn (4) is given in the Appendix.

After the determining of the unknown functions of shear force intensity in continuous connections it is possible to obtain the function of horizontal displacements of the structure as well as its derivatives using the following equations:
\[ z \in (h_{k-1}, h_k) \]
\[ V^{(k)}(z) = V_{T(k)} T_{k(k)}(z) - V_{N(k)} N_{N(k)}(z), \]  

where \( k \) is the index of a segment of the constant cross section, \( V(z) \) is the vector containing the functions of horizontal displacements of the structure, measured in the global coordinate system \( OXYZ \) and \( T_{k(k)}(z) \) is the vector of the functions of shear forces and torque resulting from lateral loads.

Matrices \( V_T, V_N \) appearing in the above relation are described by the following formulæ:
\[ V_T = (I^T K Z L)^{-1}, \quad V_N = V_T L^T C_N, \]

where \( K_z \) is the \( 3n, 3n \) matrix containing transverse stiffness of shear walls,
\[ K_z = - \text{diag}(E J_{z1}, \ldots, E J_{zn}, E J_{w1}, \ldots, E J_{wn}), \]

The boundary conditions have the following form:
\[ V_{(j)}(0) = 0, \quad V_{(j)}(0) = 0, \quad V_{(n_j)}(H) = 0. \]

Besides, at the stations, where the cross sections of the walls change, the following compatibility conditions can be stated. From the geometric compatibility consideration we have:
\[ V^{(k)}(h_k) = V^{(k+1)}(h_k), \quad V^{(k)}(h_k) = V^{(k+1)}(h_k). \]

From equilibrium consideration the following condition is obtained:
\[ m_{E(k)}(h_k) = m_{E(k+1)}(h_k), \]

where \( m_E(z) \) is the vector of bending moments and bi-moments in the shear walls, described by the relation:
\[ m_E(z) = K_z L V''(z). \]

Substituting Eqn (10) in Eqn (9) and then premultiplying by \( V_{T(k)} L^T(k) \), the following condition is obtained:
\[ V^{(k)}(h_k) = S V^{(k+1)}(h_k), \]

where:
\[ S V^{(k+1)}(h_k) = V_{T(k)} L^T(k) K_{Z(k+1)} L^T(k+1). \]

### 3. Method of solution

In the proposed method, the algorithm of solving the differential equation system, used for structures of constant cross-section [10], has been extended so as to enable taking the structures of the variable section into account.

In order to uncouple differential equation systems, auxiliary functions \( g_{ij}(z) \) satisfying these relations have been introduced:
\[ N_{N(k)}(z) = B_{j(k)}^{-1/2} Y_{i(k)} g_{i(k)}(z), \]

where \( Y_{(i)} \) is the matrix columns which are eigenvectors of the symmetrical matrix \( P_{(i)} = B_{j(k)}^{-1} A_{(i)} B_{(i)}^{-1/2} \).

Consequently, \( n_e \) second-order differential equations have been obtained in the following form:
\[ z \in (h_{k-1}, h_k) \]
\[ g_{i(k)}(z) - \lambda_i(k) g_{i(k)}(z) = F_{B(k)}, \]
\[ F_{B(k)} = Y_{i(k)} B_{i(k)}^{-1/2} f_i(z) \]

where \( \lambda_i(k) \) is the \( i \)-th eigenvalue of matrix \( P_{(k)} \), and \( Y_{i(k)} \) is the eigenvector corresponding to the \( i \)-th eigenvalue.

In the analysis, a polynomial form of functions \( f_{n}(z) \) has been used:
\[ f_{n}(z) = F_{B(i)}(z) W_{n}(z), \quad W_{n}(z) = \text{col}(z^0, \ldots, z^{n-1}). \]

The eigenvalues and eigenvectors of symmetric matrix \( P_{(k)} \) are computed by a set of procedures realizing the Householder’s tridiagonalization and the QL algorithm, which have been inserted in Ref. [13] and later written in Pascal. Matrix \( A \) is positive semi-definite, thus matrix \( P \) can also have zero eigenvalues.

The solutions of Eqn (13) corresponding to zero eigenvalues have the following form:
\[ g_{i(k)}(z) = F_{B(i)}(z) \text{col}(z^2/2, z^3/6, \ldots, z^{n+1}/(s(s+1))) + C_{i(k)} z + C_{2i(k)}. \]

The form of solutions from Eqn (13) corresponding to the non-zero eigenvalues is as follows:
\[ g_{i(k)}(z) = C_{i(k)} e^{\lambda_i(k) z} + C_{2i(k)} e^{\lambda_i(k) z} + r_{n(k)} W_{n}(z), \]

where \( C_{i(k)}, C_{2i(k)} \) are the integration constants and \( r_{n(k)} \) are particular solution coefficients, calculated by the indeterminate coefficient method.

Introducing solutions described by Eqn (15), (16) into the relation (12) and later considering boundary conditions given by Eqn (2), Eqn (3) and Eqn (4) we will obtain the system of \( 2 n_e n_e \) linear equations for the determination of all the constants of integration in the form:
\[ R_w C = P_{w}. \]

where \( R_w \) is an unsymmetric matrix and \( P_w \) is a vector dependent on the loads. The vector \( C \) successively for each segment contains: integration constants \( C_i \) corresponding to zero and non-zero eigenvalues and next integration constants \( C_j \) corresponding to the zero and non-zero eigenvalues, respectively. The solutions are computed by the procedures based on the LU factorization, where \( L \) is lower-triangular and \( U \) is upper-triangular, taken from Ref. [13].

After the determination of the integration constants \( C \), the functions of shear force intensity in continuous connections for each segment are computed in a given number of points. Then they are replaced by appropriate polynomial functions using the interpolation.

The next step of computations is the determination of functions of horizontal displacements \( V(z) \) and their derivatives necessary to calculate the internal forces and stresses.

The integration of functions \( V^{(')}(z) \) taking into consideration boundary condition \( V^{(')}(h_e)(H) = 0 \) and the compatibility condition (11) yields the following expressions:
Next, integrating the above functions with regard to boundary conditions $V_{(j)}(0) = 0$, $V_{(j)}'(0) = 0$ and compatibility conditions (8), the following is obtained:

\[
\begin{align*}
V_{(k)}(z) &= \int_{h_{k-1}}^{z} V_{(k)}(t) \, dt + S_{V(k+1,k)} V_{(k+1)}(h_{k}). \\
V_{(k)}(z) &= \int_{h_{k-1}}^{z} V_{(k)}(t) \, dt + V_{(k-1)}(h_{k-1}), \\
V_{(k)}(z) &= \int_{h_{k-1}}^{z} V_{(k)}(t) \, dt + V_{(k-1)}(h_{k}),
\end{align*}
\]

(18)

where: $k = 1, \ldots, n$, $h_0 = 0$.

In the course of determination of functions of horizontal displacements and their derivatives the polynomial form of functions $N_d(z)$ of shear force intensity in continuous connections has been used. Hence, the results may be computed for the arbitrary ordinates of height.

The derived Eqn (4) will have to be satisfied in an iterative manner. To obtain the first approximation we shall assume that the last two terms of Eqn. (4) are equal to zero. From this analysis the values of $V''_{(j)}(h_{k})$, $V''_{(j+1)}(h_{k})$, and $n_{32}(h_{k})$ can be found and then, according to Eqn (4), the improved value of the vector $P_S$ in the Eqn (17) is obtained. The analysis then carries on repeatedly, when the solution is found to be sufficiently convergent. In spite of the number of iterations required, the calculation is very fast.

Based on the presented algorithm, the software included in the system for the analysis of shear wall tall buildings [10], [11] in the Turbo Delphi from Borland Developer Studio 2006 environment has been implemented.

4. Numerical examples

While testing the program for the analysis of shear wall systems of variable cross-section there has been a good agreement of our results, those presented in Ref. [7], [8], [2], [6], [4], [1], [3], [4], [5] and those obtained from the tests on Araldite models [2]. In order to verify the algorithm for the boundary cases a number of simple examples have been prepared, for which it was possible to estimate the values of solutions. To illustrate the correctness of the algorithm realization, three examples have been chosen.

4.1. Plane wall with variable cross-section and without continuous connections

As the first example a 20-storey plane wall (Fig. 1) with an abrupt change in cross-section at the 10th storey and with stiff vertical joint in the mid-point has been analyzed.

![Figure 1: Normal stresses at the base of plane shear wall without continuous connections](image-url)
Figure 2: Horizontal displacements and shear force intensity functions in connecting beams in plane shear wall with three continuous connections.

Figure 3: Normal stresses at the base of plane shear wall with three continuous connections.
The lower and upper segments are each 50 m high, with corresponding cross-section dimensions 10 x 0.6 m and 5 x 0.6 m, respectively. The wall is subjected to a horizontal point load $P = 100$ kN, acting at the top of structure. The Young’s modulus is 30 GPa and the Kirchhoff’s modulus is 15 GPa. The horizontal displacement at the top of this structure equals to 41.67 mm. The theoretical value of the shear force in the stiff joint is 15 kN/m in segment 1 (lower) and 30 kN/m in segment 2 (upper). Maximum value of the normal stresses at the base is 1000 kPa. The computed values of displacements, shear forces and normal stresses is equal to the theoretical ones. The results for the next considered shear wall system will be compared with the results for this example. The normal stresses at the base of the structure are shown in Fig. 1.

4.2. Plane shear wall of variable cross-section with three continuous connections of small flexibility

The above described structure has been subsequently divided into the four walls each with a depth of 2.5 m, connected by three continuous connections of very small flexibility (the stiffness 2717 MN/m² has been taken). In the upper segment the left and right walls that are missing, have been taken with a depth of 0.06 mm. Introducing continuous connections of very small flexibility into the structure should result in a slight increase in the displacements. The solution converged to four significant figures in 5 iterations. The value of the horizontal displacement at the top of the structure obtained in the first iteration was 74.06 mm and the final value was 42.31 mm. Figure 2 shows the diagrams of displacements and the functions of shear force intensity in continuous connections. The vertical normal stresses at the base are shown in Fig. 3. The results were as expected and close to those obtained from previous example.

4.3. Spatial shear wall system of variable cross-section

Figure 4 shows a shear wall and shear core assembly of 30 storeys, analyzed in Ref. [4], [5]. The central core, which houses the lift shaft and the staircase, changes its geometry at the 20th floor, above which both the top-left and the bottom-right wings of the core are missing. The thickness of the core wall also varies from 0.15 m at 20th-30th floors to 0.2 m at 10th-20th floors, and finally to 0.3 m at 1st-10th floors. The thickness of the exterior plane shear walls, meanwhile, remains constant - 0.2 m. There are two types of lintel beam: those over windows having a depth of 1 m and those over doorways with a depth of 0.6 m. The storey height is 3.0 m. The Young’s modulus $E = 31$ GPa and Poisson’s ratio $\nu = 0.2$ are assumed for the concrete properties. A uniformly distributed load of 50 kN/m, acting in the Y direction, is applied along the height of the structure. The obtained horizontal displacements and distribution of shear force intensity in two bands of lintel beams are shown in Fig. 5. Figure 6 shows normal stresses at the base of the analyzed structure. In Fig. 7 there are the horizontal displacements at the top of the structure. The solution converged to four significant figures in 6 iterations. The computations correlated well with the results of the discrete force method presented in Ref. [5].
Figure 5: Horizontal displacements and shear force intensity functions in connecting beams in spatial shear wall system

Figure 6: Normal stresses at the base of the spatial shear wall system
5. Conclusions

The paper presents the algorithm for the analysis of three-dimensional shear wall structures of variable cross-section, using a variant of continuous connection method. The refined boundary conditions for derivatives of shear force intensity functions have been included. The correctness and efficiency of the continuum method is illustrated by the application of the technique in the analysis of a spatial, complex structure.

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6. References

Appendix
Derivation of boundary conditions for the functions of shear force intensity in continuous connections at each station

In the derivation of Eqn (3) and Eqn (4) presented below, the following equation, obtained on the basis of compatibility consideration at the mid-points of the cut connecting beams [12] has been used:

\[ \mathbf{B} \mathbf{N}(z) = \mathbf{C}^T \mathbf{V}(z) - \mathbf{S}^T \mathbf{V}_Z(z) \]  \hspace{1cm} (A1)

where \( \mathbf{V}(z) = \mathbf{L} \mathbf{V}(z) \) and \( \mathbf{V}_Z(z) \) is the vector containing the functions of vertical displacements of shear walls.

At the top of the \( k \)-th segment Eqn (A1) may be written as:

\[ \mathbf{B}(k) \mathbf{N}(N(k))(h_k) = \mathbf{C}^T_N \mathbf{L}(k) \mathbf{V}(k)(h_k) - \mathbf{S}^T_N \mathbf{V}_Z(k)(h_k). \]  \hspace{1cm} (A2)

In the next, \((k+1)\)-th segment, the compatibility equation (A1) may be written in the form:

\[ \mathbf{B}_{(k+1)} \mathbf{N}(N(k+1))(z) = \mathbf{C}^T_{N(k+1)} \mathbf{L}(k+1) \mathbf{V}(k+1)(z) - \mathbf{S}^T_{E(k+1)} \mathbf{V}_Z(k+1)(z) + (\mathbf{C}^T_N \mathbf{L}(k) - \mathbf{C}^T_{N(k+1)} \mathbf{L}(k+1)) \mathbf{V}(h_k). \]  \hspace{1cm} (A3)

The last term takes into account the vertical displacement of the origin of local coordinate system of shear wall in the upper, \((k+1)\)-th segment, due to a slope of the shear wall at the top of the lower, \( k \)-th segment.

Using the compatibility condition for vertical displacements of shear walls \( \mathbf{V}_Z(z) \):

\[ \mathbf{V}_Z(k)(h_k) = \mathbf{V}_Z(k+1)(h_k) \]  \hspace{1cm} (A5)

and assuming that matrix \( \mathbf{S} \) is constant for each segment, it may be noticed that right sides of Eqn (A2) and Eqn (A4) are equal. This yields the equation:

\[ \mathbf{B}(k) \mathbf{N}(N(k))(h_k) = \mathbf{B}(k+1) \mathbf{N}(N(k+1)). \]  \hspace{1cm} (A6)

By pre-multiplying Eqn (A6) by \( \mathbf{B}^{-1} \), the boundary condition, described by Eqn (3), is obtained.

To obtain the boundary condition, described by Eqn (4), the following condition for normal forces in shear walls, taken from the equilibrium consideration, is used:

\[ n_E(k)(h_k) = n_E(k+1)(h_k). \]  \hspace{1cm} (A7)

After differentiating Eqn (A1) and Eqn (A3) we get:

\[ \mathbf{B}(k) \mathbf{N}'(N(k))(h_k) = \mathbf{C}^T_N \mathbf{L}(k) \mathbf{V}'(k)(z) - \mathbf{S}^T_E \mathbf{V}'_Z(k)(z) \]  \hspace{1cm} (A8)

and

\[ \mathbf{B}(k+1) \mathbf{N}'(N(k+1))(h_k) = \mathbf{C}^T_{N(k+1)} \mathbf{L}(k+1) \mathbf{V}'(k+1)(z) - \mathbf{S}^T_E \mathbf{V}'(k+1). \]  \hspace{1cm} (A9)

The axial deformations and axial forces in shear walls are related by

\[ \mathbf{V}'(k)(z) = \mathbf{K} S(k) \mathbf{n}_E(k)(z). \]  \hspace{1cm} (A10)

Substituting Eqn (A10) in Eqn (A8) and Eqn (A9), for \( z = h_k \) the following is obtained:

\[ \mathbf{B}(k) \mathbf{N}'(N(k))(h_k) = \mathbf{C}^T_N \mathbf{L}(k) \mathbf{V}'(k)(h_k) - \mathbf{S}^T_E \mathbf{K} S(k) \mathbf{n}_E(k)(h_k) \]  \hspace{1cm} (A11)

and

\[ \mathbf{B}(k+1) \mathbf{N}'(N(k+1))(h_k) = \mathbf{C}^T_{N(k+1)} \mathbf{L}(k+1) \mathbf{V}'(k+1)(h_k) - \mathbf{S}^T_E \mathbf{K} S(k+1) \mathbf{n}_E(k)(h_k). \]  \hspace{1cm} (A12)

Subtracting Eqn (A12) from Eqn (A11), assuming that \( \mathbf{S}_{(k+1)} = \mathbf{S}_{(k)} \) and using Eqn (A7), the following is obtained:

\[ \mathbf{B}(k) \mathbf{N}'(N(k))(h_k) \mathbf{B}(k+1) \mathbf{N}'(N(k+1))(h_k) = \mathbf{C}^T_N \mathbf{L}(k) \mathbf{V}'(k)(h_k) - \mathbf{C}^T_{N(k+1)} \mathbf{L}(k+1) \mathbf{V}'(k+1)(h_k) + \mathbf{S}^T_E \mathbf{K} S(k+1) \mathbf{n}_E(k)(h_k). \]  \hspace{1cm} (A13)

By pre-multiplying each term with \( \mathbf{B}^{-1} \), the boundary condition described by Eqn (4) is obtained.